Second Harmonic Resonance on the Surface of a Magnetohydrodynamic Fluid Column

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The method of multiple scales is used to analyse the second harmonic resonance of weakly nonlinear progressive waves on the surface of a fluid column in the presence of a magnetic field. The dynamical equations governing the second harmonic resonance are obtained. Numerical results are given graphically.

1. Introduction

The stability of a cylindrical column of fluid (the 'plasma') with an axial magnetic field has often been investigated, e.g. [1, 2]. Rosenbluth [2] has, in particular, shown that when the plasma is confined between conducting walls, the presence of an axial magnetic field can, under suitable circumstances, stabilize the pinch. The simplest of the so-called *pinch configurations* consists of a cylindrical column of a fluid (the 'plasma'), inside of which a uniform axial magnetic field is present while outside there is a similar axial field together with a circumferential field falling off as inverse of the radial distance from the axis of the cylinder. In the usual arrangements, the column of fluid is circled by a concentric conducting wall. Configurations of this kind are achieved in the laboratory by sending a high current through a fluid column by means of a discharge. The axial current produces a transverse magnetic which 'pinches' the column of fluid into a configuration which is idealized in the description of the present work. The linear analysis of this problem was investigated earlier by Chandrasekhar [3].

In recent years, the evolution of wave packets on the surface of a fluid column has been investigated. Malik et al. [4] investigated the asymptotic behaviour of weakly nonlinear dispersive waves on the surface of a self-gravitating column by using the method of strained coordinates, while Chhabra and Trehan [5] discussed the effect of a uniform magnetic field on the nonlinear conditions of stability of a self-gravitating infinite fluid column by the method of multiple scales.

For a hydromagnetic column, a significant feature of the analysis is the presence of a second harmonic resonance whereby the usual analysis is not valid in its neighborhood. Various authors studied this type of resonance for a self gravitating fluid cylinder, for capillary gravity waves on deep water, and for a magnetohydrodynamic jet [6–9]. In the present paper, we consider the effects of a magnetic field on second harmonic nonlinear interactions. The phenomenon of second harmonic resonance occurs when the frequencies and wave numbers of two interacting waves satisfy the conditions $\omega_2 = 2\omega_1$ and $k_2 = 2k_1$. In this case, the fundamental and the second harmonic travel with the same phase speed. We obtain in this paper the dynamical equations involving the fundamental and second harmonics by using the multiple scale method used in [10]. The analysis of these equations shows that for certain values of the magnetic field the surface of the fluid is unstable.

2. Basic Equations

We consider axisymmetric wave motion on the surface of a fluid column whose density is ρ. Let R_0 be the radius of the fluid column and R_1 that of the encircling wall. The superscripts (1) and (2) refer to quantities inside and outside of the fluid column, respectively, and the magnetic permeability is denoted by μ. We use cylindrical coordinates (r, φ, z). $\eta(z, t)$ denotes the elevation of the free surface from the unperturbed level $r = R_0$. The motion is assumed to be irrotational. If $\boldsymbol{u}$ and $\boldsymbol{h}$ denote the velocity and magnetic field inside the fluid column, respectively, the equations holding at $r \le R_0 + \eta$ are

$$\frac{\partial \boldsymbol{h}}{\partial t} = (\boldsymbol{h} \cdot \nabla)\, \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{h}\,, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0\,, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{h}^{(1)} = 0\,, \tag{2.3}$$

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and at $r > R_0 + \eta$ we have

$$\nabla \cdot \boldsymbol{h}^{(2)} = 0 . \tag{2.4}$$

The velocity field is derived from a potential field ϕ, so that $\boldsymbol{u} = \nabla \phi$. Then the equation for ϕ is

$$\nabla^2 \phi = 0 , \quad r \le R_0 + \eta . \tag{2.5}$$

The unit normal $\boldsymbol{n}$ to the surface is given by

$$\boldsymbol{n} = \frac{\nabla F}{|\nabla F|} = -\eta_z (\eta_z^2 + 1)^{-\frac{1}{2}} \boldsymbol{e}_z + (\eta_z^2 + 1)^{-\frac{1}{2}} \boldsymbol{e}_r , \tag{2.6}$$

where $F = 0$ is the equation of the surface of the fluid column. The condition that the interface is moving with the fluid leads to

$$\eta_t - \phi_r = -\phi_z \eta_z \quad \text{at} \quad r = R_0 + \eta . \tag{2.7}$$

The normal of the magnetic field is continuous at the deformed surface of the fluid column, so that

$$\boldsymbol{n} \cdot [\![\boldsymbol{h}]\!] = 0 \quad \text{at} \quad r = R_0 + \eta , \tag{2.8}$$

where $[\![.]\!]$ represents the jump across the surface of the fluid column, i.e., $[\![h]\!] = h^{(2)} - h^{(1)}$.

At the free surface, the normal stress is continuous:

$$n_\alpha [\![p]\!] - n_\beta [\![M_{\alpha\beta}]\!] = 0 \quad \text{at} \quad r = R_0 + \eta , \tag{2.9}$$

where n_α is the unit normal vector given by (2.6), p is the pressure, and the force $M_{\alpha\beta}$ of magnetic origin is

$$M_{\alpha\beta} = \mu \left(h_\alpha h_\beta - \frac{1}{2} \delta_{\alpha\beta} h_\gamma h_\gamma \right),$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. The pressure p can be evaluated by the Bernoulli's equation. We obtain

$$p = -\frac{1}{2} \rho [\phi_r^2 + \phi_z^2] - \rho \phi_t + f(t) ,$$

where $f(t)$ stands for the constant of integration with respect to the space variables. Let H_1 and H_2 denote the strength of the axial magnetic field inside and outside of the column, respectively; and let the transverse φ-field be

$$H_\varphi = H_0 R_0 / r .$$

The fact that the magnetic field is discontinuous at $r = R_0$ means that there is a current sheet of strength

$$J_s = \frac{1}{\mu} \{ (H_1 - H_2) \boldsymbol{e}_\varphi + H_0 \boldsymbol{e}_z \}$$

on the surface. Such current sheets are features which occur when the fluid is assumed to be of infinite electrical conductivity as in the present case. Furthermore, in the stationary state the continuity of the normal stress across the surface of the fluid requires that the constant pressure p_0 inside the fluid is given by

$$p_0 = \frac{1}{2} \{ H_0^2 + H_2^2 - H_1^2 \} . \tag{2.10}$$

It will be convenient to express H_1 and H_2 in terms of H_0:

$$H_1 = \beta_1 H_0 , \quad H_2 = \beta_2 H_0 .$$

An inequality which must hold due to (2.10) is

$$1 + \beta_2^2 \ge \beta_1^2 .$$

All lengths are normalised with respect to the characteristic length R_0, the radius of the undisturbed jet, and the magnetic fields to H_0. Thus the magnetic fields inside and outside the fluid column are expressed as

$$\frac{\boldsymbol{h}^{(1)}}{H_0} = -\nabla \psi^{(1)} ,$$

$$\frac{\boldsymbol{h}^{(2)}}{H_0} = -\nabla \psi^{(2)} + \frac{1}{r} \boldsymbol{e}_\varphi .$$

Thus,

$$\nabla^2 \psi^{(1)} = 0 , \quad r \le 1 + \eta , \tag{2.11}$$

$$\nabla^2 \psi^{(2)} = 0 , \quad 1 + \eta \le r < R_1 / R_0 . \tag{2.12}$$

To investigate the modulation of a weakly nonlinear wave with a narrow band width spectrum we employ the method of multiple scales by introducing the variables

$$z_n = \varepsilon^n z \quad \text{and} \quad t_n = \varepsilon^n t \quad (n = 0, 1, 2) ,$$

and letting

$$\eta(x, t) = \sum_{n=1}^{2} \varepsilon^n \eta_n (z_0, z_1; t_0, t_1) + O(\varepsilon^3) , \tag{2.13}$$

$$\phi(z, t) = \sum_{n=1}^{2} \varepsilon^n \phi_n (z_0, z_1; t_0, t_1) + O(\varepsilon^3) , \tag{2.14}$$

$$\psi^{(j)}(z, t) = \sum_{n=0}^{2} \varepsilon^n \psi_n^{(j)} (z_0, z_1; t_0, t_1) + O(\varepsilon^3) , \quad (j = 1, 2) , \tag{2.15}$$

where the small parameter ε characterizes the steepness ratio of the wave. The expansions (2.13) to (2.15) are assumed to be uniformly valid for $-\infty < z < \infty$ and $0 < t < \infty$. The quantities appearing in (2.1) to (2.5) and the boundary conditions (2.7) to (2.9) can now be expressed in Maclaurin Series expansions around $r=1$. Then, we use (2.13) to (2.15) and equate the coefficients of equal powers in ε to obtain the linear and successive nonlinear partial differential equations of various orders. The hierarchy of equation for each order can be derived with the knowledge of the solutions for the previous order.

3. Linear Theory

Substituting the expansions given by (2.13) to (2.15) into the field equations (2.1) to (2.5) and boundary conditions (2.7) to (2.9), and equating terms of equal powers of ε on both sides of the equations we obtain the following set of equations and boundary conditions to order ε. The zeroth order solution yields

$$\psi_0^{(1)} = -\beta_1 z_0, \quad \psi_0^{(2)} = -\beta_2 z_0. \tag{3.1}$$

Inside of the fluid column we have to solve

$$\nabla_0^2 \phi_1 = 0, \tag{3.2}$$

$$\nabla_0^2 \psi_1^{(1)} = 0, \tag{3.3}$$

$$\frac{\partial \psi_1^{(1)}}{\partial t_0} + \beta_1 \frac{\partial \phi_1}{\partial z_0} = 0. \tag{3.4}$$

Outside of the fluid column we solve

$$\nabla_0^2 \psi_1^{(2)} = 0. \tag{3.5}$$

The boundary conditions are

$$\left[\!\left[\frac{\partial \psi_1}{\partial r} \right]\!\right] - \frac{\partial \eta_1}{\partial z_0} \left[\!\left[\frac{\partial \psi_0}{\partial z_0} \right]\!\right] = 0 \quad \text{at} \quad r=1, \tag{3.6}$$

$$\frac{\partial \eta_1}{\partial t_0} - \frac{\partial \phi_1}{\partial r} = 0 \quad \text{at} \quad r=1, \tag{3.7}$$

$$-\frac{\partial \phi_1}{\partial t_0} + \alpha \eta_1 + \alpha \left[\!\left[\beta \frac{\partial \psi_1}{\partial z_0} \right]\!\right] = 0 \quad \text{at} \quad r=1, \tag{3.8}$$

where

$$\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_0^2}, \tag{3.9}$$

and α is the magnetic parameter defined by $\alpha = R_0 H_0^2 \mu$ with the solutions

$$\eta_1 = A(z_1, t_1)\exp(i\theta) + c.c., \tag{3.10}$$

$$\phi_1 = -i\frac{\omega}{k}\frac{I_0(kr)}{I_1(k)} A(z_1, t_1)\exp(i\theta) + c.c., \tag{3.11}$$

$$\psi_1^{(1)} = -i\beta_1 \frac{I_0(kr)}{I_1(k)} A(z_1, t_1)\exp(i\theta) + c.c., \tag{3.12}$$

$$\psi_1^{(2)} = i\beta_2 \frac{K_0(kr)}{K_1(k)} A(z_1, t_1)\exp(i\theta) + c.c., \tag{3.13}$$

where

$$\theta = k z_0 - \omega t_0. \tag{3.14}$$

Here, k and ω stand for the wavenumber and the frequency, respectively. The progressive solutions (3.10) – (3.13) lead to the dispersion relation

$$D(\omega, k) = -\frac{k I_1(k)}{I_0(k)} \cdot \alpha \left[1 - \beta_2^2 k \frac{K_0(k)}{K_1(k)} - \beta_1^2 k \frac{I_0(k)}{I_1(k)} \right] - \omega^2. \tag{3.15}$$

We now investigate the conditions under which the two waves can interact resonantly. For the occurrence of such a resonant interaction, it is necessary that (k, ω) and $(nk, n\omega)$ for some integer greater than $n=1$ must depend upon each other through the dispersion relation (3.15). The first resonant wave number k_1, corresponding to $n=2$ is given by the equation

$$\frac{2}{I_a}[1 - \beta_2^2 k_1 K_a - \beta_1^2 k_1 I_a] = \frac{1}{I_b}[1 - \beta_2^2 2k_1 K_b - \beta_1^2 2k_1 I_b], \tag{3.16}$$

where

$$K_a = \frac{K_0(k_1)}{K_1(k_1)}, \quad I_a = \frac{I_0(k_1)}{I_1(k_1)}, \quad K_b = \frac{K_0(2k_1)}{K_1(2k_1)}, \quad I_b = \frac{I_0(2k_1)}{I_1(2k_1)}. \tag{3.17}$$

The roots of (3.16) can be obtained for various values of β_1 and β_2 and are shown in Figure 1.

In order to describe resonant interaction at or in the neighborhood of k_1, we write

$$\eta_1 = \sum_{m=1}^{2} A_m(z_1, t_1)\exp(i\theta_m) + c.c., \tag{3.18}$$

$$\phi_1 = -i \sum_{m=1}^{2} \frac{\omega_m}{k_m}\frac{I_0(k_m r)}{I_1(k_m)} \cdot A_m(z_1, t_1)\exp(i\theta_m) + c.c., \tag{3.19}$$

$$\psi_1^{(1)} = -i\beta_1 \sum_{m=1}^{2} \frac{I_0(k_m r)}{I_1(k_m)} \cdot A_m(z_1, t_1)\exp(i\theta_m) + c.c., \tag{3.20}$$

$$\psi_1^{(2)} = i\beta_2 \sum_{m=1}^{2} \frac{K_0(k_m r)}{K_1(k_m)} \cdot A_m(z_1, t_1)\exp(i\theta_m) + c.c., \tag{3.21}$$

where

$$\theta_m = k_m z_0 - \omega_m t_0, \tag{3.22}$$

$$\omega_m^2 = -\frac{k_m I_1(k_m)}{I_0(k_m)} \cdot \alpha\left[1 - \beta_2^2 k_m \frac{K_0(k_m)}{K_1(k_m)} - \beta_1^2 k_m \frac{I_0(k_m)}{I_1(k_m)}\right], \tag{3.23}$$

$$k_2 = 2k_1, \quad \omega_2 = 2\omega_1, \quad \text{and} \quad \theta_2 = 2\theta_1. \tag{3.24}$$

4. Second Order Solutions

We now proceed to the second order problem in $O(\varepsilon^2)$. With the use of the first order solutions given by (3.18) to (3.21), we obtain the equations for the second order problem:

$$\nabla_0^2 \phi_2 = -2 \sum_{m=1}^{2} \omega_m \frac{I_0(k_m r)}{I_1(k_m)} \frac{\partial A_m}{\partial z_1} \cdot \exp(i\theta_m) + c.c., \tag{4.1}$$

$$\nabla_0^2 \psi_2^{(1)} = -2\beta_1 \sum_{m=1}^{2} k_m \frac{I_0(k_m r)}{I_1(k_m)} \frac{\partial A_m}{\partial z_1} \cdot \exp(i\theta_m) + c.c., \tag{4.2}$$

$$\frac{\partial \psi_2^{(1)}}{\partial t_0} + \beta_1 \frac{\partial \phi_2}{\partial z_0} = i\beta_1 \sum_{m=1}^{2} \left\{ \frac{\omega_m}{k_m} \frac{I_0(k_m r)}{I_1(k_m)} \frac{\partial A_m}{\partial z_1} + \frac{I_0(k_m r)}{I_1(k_m)} \frac{\partial A_m}{\partial t_1} \right\} e^{i\theta_m} + c.c., \tag{4.3}$$

$$\nabla_0^2 \psi_2^{(2)} = 2\beta_2 \sum_{m=1}^{2} k_m \frac{K_0(k_m r)}{K_1(k_m)} \frac{\partial A_m}{\partial z_1} \cdot \exp(i\theta_m) + c.c.. \tag{4.4}$$

Equations (4.1) to (4.4) furnish the second order solutions:

$$\phi_2 = \sum_{m=1}^{2} \left\{ -\frac{\omega_m}{k_m} \frac{r I_1(k_m r)}{I_1(k_m)} \frac{\partial A_m}{\partial z_1} + B_m \frac{I_1(k_m)}{I_0(k_m)} \right\} e^{i\theta_m} + c.c., \tag{4.5}$$

$$\psi_2^{(1)} = \beta_1 \sum_{m=1}^{2} \left[\left\{ -\frac{r I_1(k_m r)}{I_1(k_m)} - \frac{I_0(k_m r)}{k_m I_1(k_m)} \right\} \cdot \frac{\partial A_m}{\partial z_1} - \frac{1}{\omega_m} \frac{I_0(k_m r)}{I_1(k_m)} \frac{\partial A_m}{\partial t_1} + B_m \frac{k_m}{\omega_m} \frac{I_0(k_m r)}{I_1(k_m)} \right] e^{i\theta_m} + c.c., \tag{4.6}$$

$$\psi_2^{(2)} = -\beta_2 \sum_{m=1}^{2} \left\{ \frac{r K_1(k_m r)}{K_1(k_m)} \frac{\partial A_m}{\partial z_1} + C_m \frac{K_0(k_m r)}{K_1(k_m)} \right\} e^{i\theta_m} + c.c., \tag{4.7}$$

$$\eta_2 = \sum_{m=1}^{2} D_m \exp(i\theta_m) + c.c.. \tag{4.8}$$

The kinematical boundary condition at $r=1$ is

$$-\frac{\partial \eta_2}{\partial t_0} + \frac{\partial \phi_2}{\partial r} = \frac{\partial \eta_1}{\partial t_1} - \eta_1 \frac{\partial^2 \phi_2}{\partial r^2} + \frac{\partial \phi_1}{\partial z_0} \frac{\partial \eta_1}{\partial z_0}. \tag{4.9}$$

When the terms of the form $\varepsilon^2 e^{i\theta_1}$ are considered we obtain

$$D_1 i\omega_1 + B_1 k_1 = \frac{\partial A_1}{\partial t_1} + \omega_1 I_a \frac{\partial A_1}{\partial z_1} + A_2 A_1^* i\omega_1 (k_1 I_a + k_2 I_b - 1) + c.c., \tag{4.10}$$

where * denotes complex conjugate. Similarly, when terms of the form $\varepsilon^2 e^{i\theta_2}$ are considered we obtain

$$D_2 i\omega_2 + B_2 k_2 = \frac{\partial A_2}{\partial t_1} + \omega_2 I_b \frac{\partial A_2}{\partial z_1} + A_1^2 i\omega_1 (2k_1 I_a - 1) + c.c.. \tag{4.11}$$

The boundary condition for the magnetic potentials at $r=1$ is

$$\frac{\partial \eta_1}{\partial z_0}\left[\!\left[\frac{\partial \psi_1}{\partial z_0} \right]\!\right] + (\beta_1 - \beta_2)\left\{ \frac{\partial \eta_1}{\partial z_1} + \frac{\partial \eta_2}{\partial z_0} \right\} - \left[\!\left[\frac{\partial \psi_2}{\partial r} - \frac{\partial^2 \psi_1}{\partial r^2} \eta_1 \right]\!\right] = 0. \tag{4.12}$$

From (4.12) when the terms of the form $\varepsilon^2 e^{i\theta_1}$ are considered we obtain

$$(\beta_1-\beta_2)\,ik_1\,D_1-\beta_2\,k_1\,C_1+\beta_1\,\frac{k_1^2}{\omega_1}\,B_1$$
$$=\frac{\partial A_1}{\partial z_1}\,\{k_1\,(\beta_2\,K_a+\beta_1\,I_a)+\beta_2\}+\beta_1\,\frac{k_1}{\omega_1}\,\frac{\partial A_1}{\partial t_1}$$
$$+\,A_2\,A_1^*\,i\,k_1\,\{4\,k_1\,(\beta_2\,K_b+\beta_1\,I_b) \qquad (4.13)$$
$$+\,k_1\,(\beta_2\,K_a+\beta_1\,I_a)+\beta_2-\beta_1\}\,.$$

Similarly, when the terms of the form $\varepsilon^2 e^{i\theta_2}$ are considered we obtain

$$(\beta_1-\beta_2)\,ik_2\,D_2-\beta_2\,k_2\,C_2+\beta_1\,\frac{k_2^2}{\omega_2}\,B_2$$
$$=\frac{\partial A_2}{\partial z_1}\,\{k_2\,(\beta_2\,K_b+\beta_1\,I_b)+\beta_2\}+\beta_1\,\frac{k_2}{\omega_2}\,\frac{\partial A_2}{\partial t_1}$$
$$+\,A_1^2\,i\,k_1\,\{2\,k_1\,(\beta_2\,K_a+\beta_1\,I_a)+\beta_2-\beta_1\}\,. \qquad (4.14)$$

Now, condition (2.9) is

$$-\frac{\partial\phi_2}{\partial t_0}+\alpha\,\eta_2+\alpha\left[\!\!\left[\beta\,\frac{\partial\psi_2}{\partial z_0}\right]\!\!\right]=\frac{\partial\phi_1}{\partial t_1}+\eta_1\,\frac{\partial^2\phi_1}{\partial t_1\,\partial r}$$
$$+\frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial r}\right)^2+\left(\frac{\partial\phi_1}{\partial z_0}\right)^2\right]+\frac{3}{2}\,\alpha\,\eta_1^2 \qquad (4.15)$$
$$-\,\alpha\left[\!\!\left[\frac{1}{2}\left(\frac{\partial\psi_1}{\partial r}\right)^2+2\,\beta\,\frac{\partial\phi_1}{\partial r}\,\frac{\partial\eta_1}{\partial z_0}+\beta^2\left(\frac{\partial\eta_1}{\partial z_0}\right)^2\right.\right.$$
$$\left.\left.+\,\beta\left(\frac{\partial\psi_1}{\partial z_1}+\frac{\partial^2\psi_1}{\partial r\,\partial z_0}\,\eta\right)-\frac{1}{2}\left(\frac{\partial\psi_1}{\partial z_0}\right)^2\right]\!\!\right]\,.$$

Substituting second order solutions into (4.15) and eliminating B_m, C_m, and D_m $(m=1,2)$ using (4.10)–(4.11) and (4.13)–(4.14), we obtain dynamical equations for the coupled amplitude

$$\frac{\partial A_1}{\partial t_1}+\frac{\partial A_1}{\partial z_1}\,u_1=i\,A_2\,A_1^*\,q_1\,, \qquad (4.16)$$

$$\frac{\partial A_2}{\partial t_1}+\frac{\partial A_2}{\partial z_1}\,u_2=i\,A_1^2\,q_2\,, \qquad (4.17)$$

where

$$q_1=-\frac{k_1}{2\,\omega_1\,I_a}\Big[\omega_1^2\,(-3+2\,I_a\,I_b)$$
$$-\,\alpha\,\{\beta_2^2\,k_1^2\,(3-2\,K_a\,K_b)-\beta_1^2\,k_1^2\,(3+2\,I_a\,I_b)-3\}$$
$$+\,I_a\,(\omega_1^2-\alpha\,k_1^2\,\beta_1^2)\left(4\,K_b+2\,I_b+K_a+I_a+\frac{2\,\beta_1}{\beta_2}\,I_b\right)$$
$$+\,\alpha\left(4\,k_1\,K_b+k_1\,K_a+1+2\,k_1\,\frac{\beta_1}{\beta_2}\,I_b\right)\Big]\,, \qquad (4.18)$$

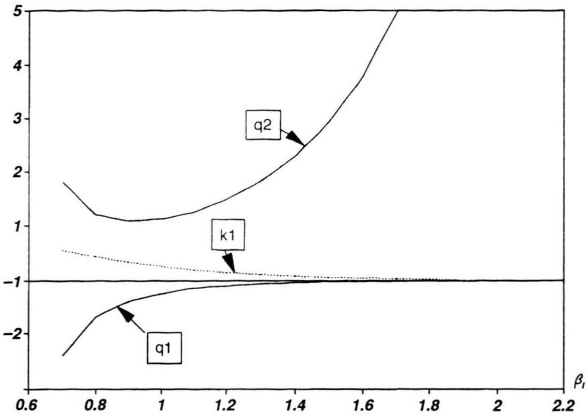

Fig. 1. Graphs of k_1 and the first resonant wave numbers, q_1 and q_2 as functions of the magnetic field.

$$q_2=-\frac{k_2}{2\,\omega_2\,I_b}\Big[2\,I_b\,(\omega_1^2-\alpha\,k_1^2\,\beta_1^2)\,(K_a+I_a)$$
$$+\frac{1}{2}\,(2\,k_1\,K_a+1)+\frac{1}{2}\,(I_a^2-3)\,\omega_1^2$$
$$+\,\alpha\,\frac{k_1^2}{2}\,\{\beta_2^2\,(K_a^2-3)-\beta_1^2\,(I_a^2-3)\}\Big]\,, \qquad (4.19)$$

$$u_m=\frac{\mathrm{d}\omega_m}{\mathrm{d}k_m}=\frac{k_m}{2\,\omega_m\,I_m}\left[\frac{\omega_m^2}{k_m}\,I_m\left(I_m-\frac{1}{I_m}+\frac{2}{k_m^2}\right)\right.$$
$$\left.+\,\alpha\left\{\frac{2}{k_m}+\beta_2^2\,k_m\,(K_m^2-1)-\beta_1^2\,k_m\,(I_m^2-1)\right\}\right]\,,$$
$$(m=1,2)\,, \qquad (4.20)$$

with

$$I_1=I_a\,,\quad I_2=I_b\,,\quad K_1=K_a\,,\quad K_2=K_b\,. \qquad (4.21)$$

We have calculated the values of q_1 and q_2 for various values of β_1 when $\beta_1=\beta_2$. They are shown in Figure 1.

5. Steady State Solutions

We look for solutions of the type

$$A_m=a_m(\xi)\,e^{i\varphi_m(\xi)}\,,\quad (m=1,2) \qquad (5.1)$$

with $\xi=t_1-\lambda z_1$ and λ being a real constant. We substitute from (5.1) into (4.16) and (4.17). Then, on separation of real and imaginary parts, (4.16) and (4.17) become

$$a_1'=\frac{a_1\,a_2}{\alpha_1}\,\sin\varphi\,, \qquad (5.2)$$

$$\varphi_1' = \frac{a_2 \cos \varphi}{\alpha_1}, \tag{5.3}$$

$$a_2' = -\frac{a_1^2}{\alpha_2} \sin \varphi, \tag{5.4}$$

$$\varphi_2' = \frac{a_1^2 \cos \varphi}{\alpha_2 a_2}, \tag{5.5}$$

where

$$\alpha_m = \frac{1 - \lambda u_m}{q_m}, \quad (m = 1, 2), \tag{5.6}$$

and

$$\varphi = 2\varphi_1 - \varphi_2 . \tag{5.7}$$

Equations (5.2) and (5.4) lead to

$$a_1^2 + \frac{\alpha_2}{\alpha_1} a_2^2 = E_0 / \alpha_1, \tag{5.8}$$

where E_0 is a constant of integration. If α_1 and α_2 have the same sign, (5.8) implies that a_1 and a_2 are always bounded. However, if α_1 and α_2 have opposite sign, a_1 and a_2 may grow with time even though $a_1^2 - |\alpha_2/\alpha_1| a_2^2$ is bounded. From (5.2)–(5.5) (devide (5.2) by (5.3), (5.4) by (5.5), rearrange, add and integrate) we can easily obtain

$$a_1^2 a_2 \cos \varphi = L, \tag{5.9}$$

where L is a constant of integration. Equations (5.2), (5.8), and (5.9) lead to

$$a_1'^2 + \frac{L^2}{a_1^2 \alpha_1^2} - \frac{E_0}{\alpha_1^2 \alpha_2} + \frac{a_1^4}{\alpha_1 \alpha_2} = 0 . \tag{5.10}$$

It is more convenient to put $1/2\, a_1^2 = E_1$. Then (5.10) along with the substitution reduces to

$$E_1'^2 + \frac{L^2}{\alpha_1^2} - \frac{8 E_1^2}{\alpha_1 \alpha_2} \left(\frac{E_0}{2\alpha_1} - E_1 \right) = 0 . \tag{5.11}$$

In view of the definition of $E_1 \geq 0$, and also from (5.8) and (5.9) we must have $E_1 \leq E_0 / 2\alpha_1$. Therefore the solution of (5.10) exists only if

$$E_1'^2 \geq 0, \quad \alpha_1 \alpha_2 > 0, \quad 0 \leq L^2 \leq \frac{4 E_0^3}{27 \alpha_1^2 \alpha^2} . \tag{5.12}$$

The values of λ for which $\alpha_2 \alpha_1^{-1} > 0$ were calculated for various values of β_1 and β_2 and are shown in Figs. 2 and 3. The motion is bounded when λ lies within the shaded region.

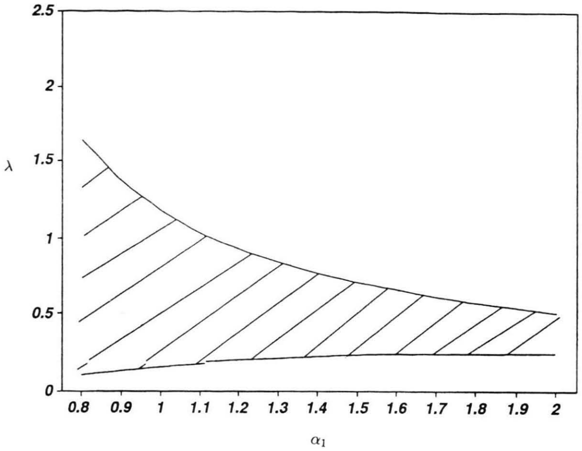

Fig. 2. λ vs. α_1 for $\alpha_1 = \alpha_2$.

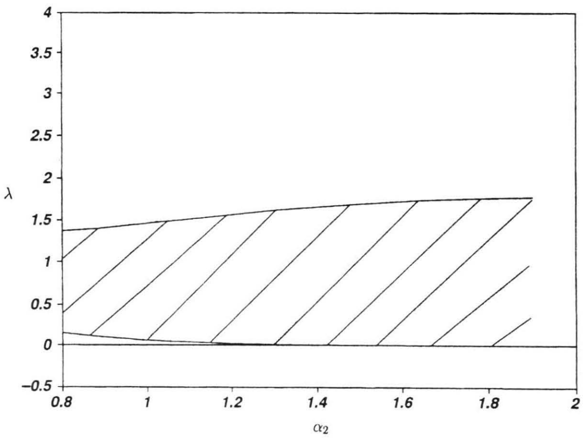

Fig. 3. λ vs. α_2 when $\alpha_1 = 0.9$.

The solution of (5.11) can be obtained using elliptic functions. We write (5.11) as

$$\frac{\alpha_1 \alpha_2}{8} \left(\frac{dE_1}{dt_1} \right)^2 = (e_3 - E_1)(E_1 - e_2)(E_1 - e_1), \tag{5.13}$$

where $e_1 \leq e_2 \leq e_3$. Let

$$e_3 - E_1 = (e_3 - e_2) \sin^2 \chi . \tag{5.14}$$

Now (5.13) along with (5.14) reduces to

$$\sqrt{\frac{\alpha_1 \alpha_2}{2}} \frac{d\chi}{dt_1} = \pm \sqrt{e_3 - e_1}\, (1 - S^2 \sin^2 \chi)^{1/2}, \tag{5.15}$$

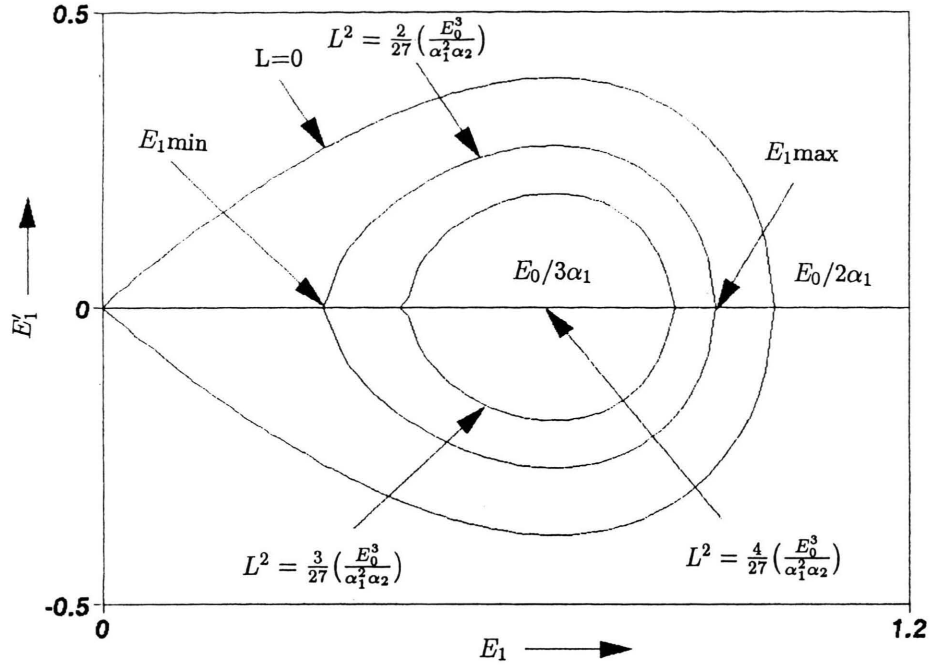

Fig. 4. $E_1 - E_1'$ plane for fixed E_0; $0 \le L^2 \le \dfrac{4E_0^3}{27\alpha_1^2\alpha_2}$.

where

$$S = \sqrt{\frac{e_3 - e_2}{e_3 - e_1}} .$$

The solution of (5.15) is expressed as

$$\sin\chi = \mathrm{sn}\,[\Xi; S] , \tag{5.16}$$

where sn is a Jacobian elliptic function and

$$\Xi = \varepsilon\sqrt{2\frac{e_3 - e_1}{\alpha_1\alpha_2}}\,(t_1 - t_0) ,$$

where $t_1 = \varepsilon t$, and εt_0 corresponds to $\chi = 0$. Equations (5.14) and (5.16) reduce to

$$E_1 = e_3 - (e_3 - e_2)\,\mathrm{sn}^2[\Xi; S] , \tag{5.17}$$

A graphic presentation of the amplitude modulations, devoid of the cumbersome elliptic notation is most easily presented in the phase planes for the individual amplitude (Fig. 4) which are easily constructed from (5.11) directly.

The outermost trajectories are those for which $L=0$, for which the individual phases are constant and the relative phase is exactly $\pm 1/2\,\pi$. The modulation period becomes infinite, and the solutions degenerate to their limiting cases of hyperbolic functions. The existence of this solution requires very special conditions: namely, if the relative phase is ever $\pm 1/2\,\pi$, then it remains so for all time.

For initial conditions such that $0 < L^2 < 2/27\,(E_0^3/\alpha_1^2\alpha_2)$, which corresponds to the inner trajectories in Fig. 4, the picture is one of simultaneous amplitude and phase modulations. The individual amplitudes can never vanish, and the phases are frequency modulated about a slightly shifted mean frequency.

The remaining possibility is for initial conditions such that $L = (4E/27\alpha_1^2\alpha_2)^{1/2}$. For nonzero amplitudes this requires that $\cos\varphi = 1$, whence the relative phase modulation vanishes. The trajectories become the single points on the abscissae of Fig. 4; that is the amplitude modulations vanish entirely. The solutions become $a_1^2 = 2E_0/3\alpha_1$, $a_2^2 = E_0/3\alpha_2$.

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